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**COURSE: THEORETICAL MECHANICS (MA 278)**

**ASSIGNMENT 1**

1. Prove that the total external torque on a system of particles is equal to the time rate of change of the angular momentum of the system, provided the internal forces between the particles are central forces.
2. Show that the force field **F** defined by $\boldsymbol{F} = (y^2z^3 - 6xz^2)\boldsymbol{i} + (2xyz^3)\boldsymbol{j} + (3xy^2z^2 - 6x^2z)\boldsymbol{k}$ is a conservative force field.
3. A mass of $5000kg$ moves on a straight line from a speed of $540km/h$ to $720km/h$ in2 minutes. What is the impulse developed in this time?
4. A particle of mass 2 moves in a force field depending on time $t$ given by $\boldsymbol{F} = 24t^2\boldsymbol{i} + (36t - 16)\boldsymbol{j} + 12t\boldsymbol{k}$. Assuming that at $t = 0$ the particle is located at $r_0 = 3\boldsymbol{i} - \boldsymbol{j} + 4\boldsymbol{k}$ and has a velocity$v_0 = 6i + 15j - 8k$, find the

(a) velocity

(b) position at any time $t$

(c) torque

(d) angular momentum about the origin for the particle at any time $t$.

ANSWER

**1.** The moment of force or torque about the origin $O$ is

$\Lambda = r \times F$ ………….. **1**

where $F = \frac{d}{dt}(mv)$ ………….. **2**

substitute eqn **2** into eqn **1**

Torque, $\Lambda = r \times \frac{d}{dt}(mv)$ ………….. **3**

Also, the angular momentum or moment of momentum about $O$ is,

$\Omega = m(r \times v) = r \times (mv)$

Now we have, $\frac{d\Omega}{dt} = r \times \frac{d}{dt}(mv)$ ………….. **4**

substitute eqn **4** into eqn **3**

Torque, $\Lambda = \frac{d\Omega}{dt}$ as required.

**2.** Given that, $\boldsymbol{F} = (y^2z^3 - 6xz^2)\boldsymbol{i} + (2xyz^3)\boldsymbol{j} + (3xy^2z^2 - 6x^2z)\boldsymbol{k}$

To show the force, $F$ is conservative force field. The force field $F$ is conservative if and only if $curl\ F = \nabla \times F = 0$

Now, $\nabla = \frac{d}{dx}i + \frac{d}{dy}j + \frac{d}{dz}k$

$$\nabla \times F = \begin{vmatrix} i & j & k \\ \frac{d}{dx} & \frac{d}{dy} & \frac{d}{dz} \\ y^2z^3 - 6xz^2 & 2xyz^3 & 3xy^2z^2 - 6x^2z \end{vmatrix}$$

$$i\left[\frac{d}{dy}(3xy^2z^2 - 6x^2z) - \frac{d}{dz}(2xyz^3)\right] - j\left[\frac{d}{dx}(3xy^2z^2 - 6x^2z) - \frac{d}{dz}(y^2z^3 - 6xz^2)\right]$$

$$+ k\left[\frac{d}{dx}(2xyz^3) - \frac{d}{dy}(y^2z^3 - 6xz^2)\right]$$

$\nabla \times F = i[6xyz^2 - 6xyz^2] - j[3y^2z^2 + 12xz - 3y^2z^2 + 12xz] + k[2yz^3 - 2yz^3]$

$\nabla \times F = 0 - 0 + 0 = 0$

Hence the force, $F$ is a conservative force.

**3.** Given mass, $m = 5000kg$ and speed $v_1 = 540km/h$ and $v_2 = 720km/h$

Assume that the mass travels in the direction of the positive $x - axis$ in the $m/s$ system.

$km$ $540i \times 1000m$

$v_1 = 540i \frac{}{hr} \frac{}{3600\ sec} =$

$v_1 = 1.5 \times 10^2 i\ m/s$

Also,

$v_2 = 720i \frac{km}{hr} \frac{720i \times 1000m}{3600\ sec} =$

$v_2 = 2.0 \times 10^2 i\ m/s$

Impulse $= m(v_2 - v_1)$

$= 5000 \times (2.0 \times 10^2 i - 1.5 \times 10^2 i\ ) m/s \times kg$

$= 5000kg \times 0.5 \times 10^2 i\ m/s$

Impulse $= 2.5 \times 10^2 i$ *newton sec* in the positive $x$ direction.

**4.** Given mass $m = 2kg$, force $F = 24t^3 i + (36t - 16)j + 12tk$ where $r_0 = 3i - j + 4k$ and

$v_0 = 6i + 15j - 8k$

a) From newton's second law, $F = ma$

$$F = m \frac{dv}{dt}$$

$$24t^3 i + (36t - 16)j + 12tk = 2\frac{dv}{dt}$$

$$\frac{dv}{dt} 12t^3 i + (18t - 8)j + 6tk =$$

Integrating with respect to $t$ and calling $c_1$, the constant of integration we have,

$$\int dv = \int 12t^3 i + (18t - 8)j + 6tk\ dt$$

$$v = 3t^4 i + (9t^2 - 8t)j + 3t^2 k + c_1$$

Since $v = v_0 = 6i + 15j - 8k$ at $t = 0$ we have,

$c_1 = 6i + 15j - 8k$

And so, velocity

$v = 3t^4 i + (9t^2 - 8t)j + 3t^2 k + 6i + 15j - 8k$

$v = (3t^4 + 6)i + (9t^2 - 8t + 15)j + (3t^2 - 8)k$

b) Since, velocity $v = \frac{dr}{dt}$

from part (a)

$$\frac{dr}{dt} = (3t^4 + 6)i + (9t^2 - 8t + 15)j + (3t^2 - 8)k$$

Integrating with respect to $t$ and calling $c_2$, the constant of integration we have,

$$\int dr = \int [(3t^4 + 6)i + (9t^2 - 8t + 15)j + (3t^2 - 8)k]dt$$

$$r = (\frac{3}{5}t^5 + 6t)\, i + (3t^3 - 4t^2 + 15t)j + (t^3 - 8t)k + c_2$$

Since $r = r_0 = 3i - j + 4k$ at $t = 0$ we have,

$c_2 = 3i - j + 4k$

The position vector at any time $t$;

$$r = (\frac{3}{5}t^5 + 6t)\, i + (3t^3 - 4t^2 + 15t)j + (t^3 - 8t)k + 3i - j + 4k$$

$$r = (\frac{3}{5}t^5 + 6t + 3)i + (3t^3 - 4t^2 + 15t - 1)j + (t^3 - 8t + 4)k$$

c) Torque, $\Lambda = r \times F$

$$\Lambda = [(\frac{3}{5}t^5 + 6t + 3)\, i + (3t^3 - 4t^2 + 15t - 1)j + (t^3 - 8t + 4)k] \times$$

$[24t^3 i + (36t - 16)j + 12tk]$

$i \qquad j \qquad k$

$$= \begin{vmatrix} (\frac{3}{5}t^5 + 6t + 3) & (3t^3 - 4t^2 + 15t - 1) & (t^3 - 8t + 4) \\ 24t^3 & (36t - 16) & 12t \end{vmatrix}$$

$i[12t(3t^3 - 4t^2 + 15t - 1) - (36t - 16)(t^3 - 8t + 4)] -$

$j\,[12t\,(\frac{3}{5}t^5 + 6t + 3) - 24t^3(t^3 - 8t + 4)] +$

$k\,[(36t - 16)(\frac{3}{5}t^5 + 6t + 3) - 24t^3(3t^3 - 4t^2 + 15t - 1)]$

By expanding terms and groping like terms the result obtained is;

$$torque = (-32t^3 + 468t^2 - 284t + 64)i + \frac{(84t^5 - 192t^4 + 96t^3 - 72t^2 - 36t)}{5}j$$

$(-32t^3 + 468t^2 - 284t + 64)i + \frac{(84t^5 - 192t^4 + 96t^3 - 72t^2 - 36t)}{5}j$, as the torque of the particles.

d) Finding the angular momentum, let $L = r \times \rho$ , where $\rho = mv$

$\Rightarrow L = r \times mv$

$L = m(r \times v)$, where

$r = (\frac{3}{5}t^5 + 6t + 3)i + (3t^3 - 4t^2 + 15t - 1)j + (t^3 - 8t + 4)k$

And $v = (3t^4 + 6)i + (9t^2 - 8t + 15)j + (3t^2 - 8)k$

$$L = \begin{vmatrix} i & j & k \\ (\frac{3}{5}t^5 + 6t + 3) & (3t^3 - 4t^2 + 15t - 1) & (t^3 - 8t + 4) \\ (3t^4 + 6) & (9t^2 - 8t + 15) & (3t^2 - 8) \end{vmatrix}$$

By expanding terms and grouping like terms the result obtained is;

Angular moment$(L)$ =

$(-8t^4 + 15t^3 - 142t^2 + 64t - 104)i$

$$+\left(-\frac{12}{5}t^7 - t^5 + \frac{192}{5}\,24t^4 - 24t^3 - 18t^2 + 96\right) j$$

$$+ \quad \frac{\phantom{0}}{5} \quad \frac{\phantom{0}}{5}\left(-36\,t^7 + 72\,t^6 - 72t^5 + 6t^4 + 72t^3 + 6t^2 - 48t\right.$$

$\left. + 102\right)k$ As the angular momentum of the particles